# The Ising model for magnets and the mysterious Lee-Yang zeros.

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IUPUI

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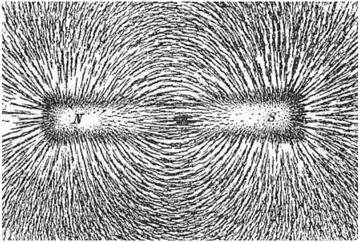

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2. Convince you that there are interesting open problems about the Ising Model.
3. Convince you that working on research problems involving many different types of math is valuable and rewarding.

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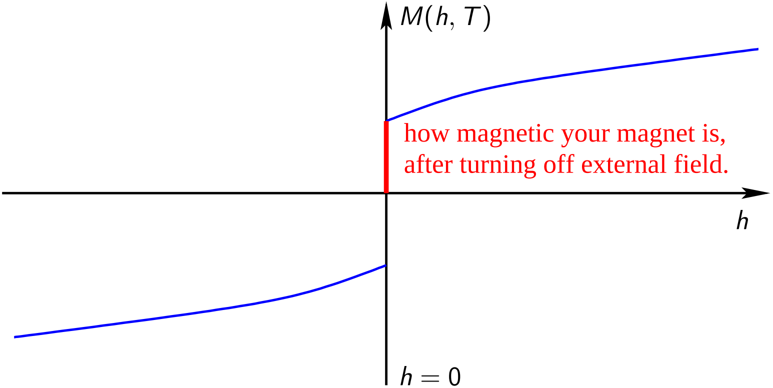
Note that if the temperature is too high (above 1420°F for iron) then this doesn't work.

# Phenomenology of magnets, $T < T_c$




$h$ is the external magnetic field applied to your piece of iron.
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$M(h, T)$ is how magnetic your piece of iron is (magnetization).

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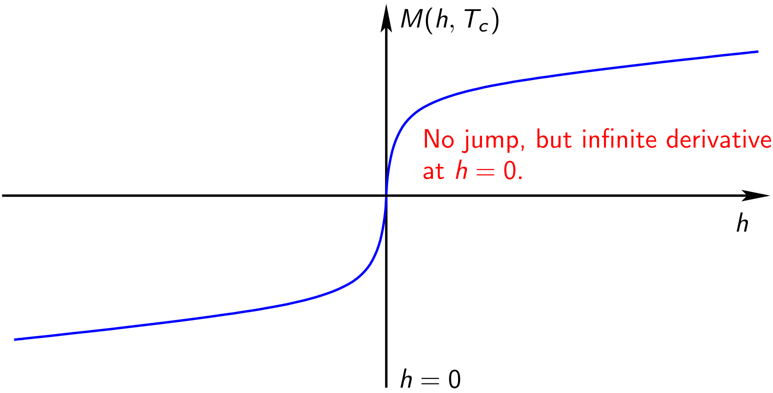


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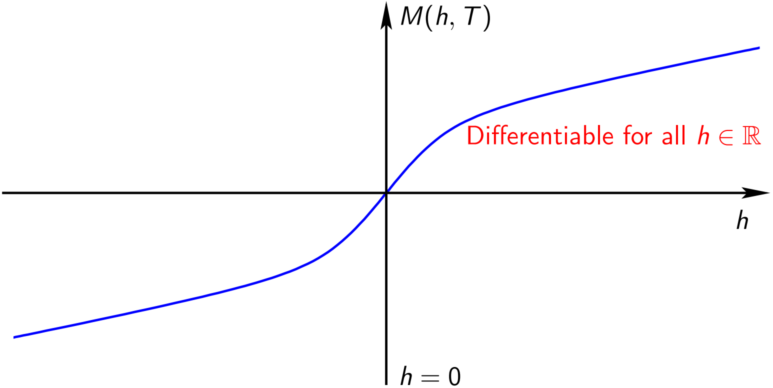


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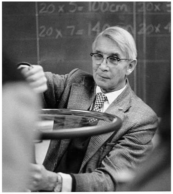
Lentz

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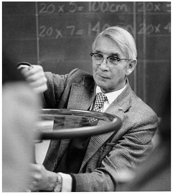

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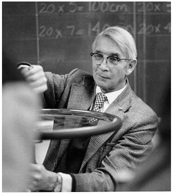

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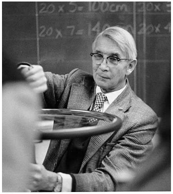


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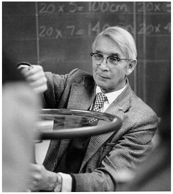

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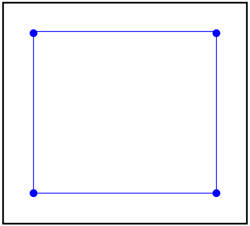

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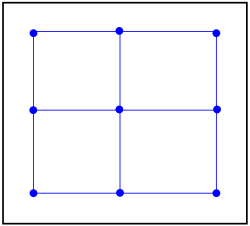

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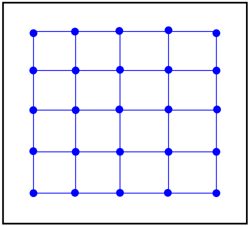

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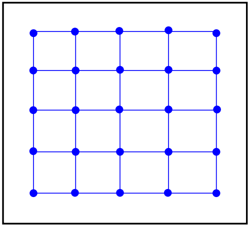

For any configuration of spins $\sigma : V_n \to \{\pm 1\}$, the system has energy

$$H_n(\sigma) = -J \sum_{(v,w)\in E_n} \sigma(v)\sigma(w) - h \sum_{v \in V_n} \sigma(v),$$

where $J > 0$ is a coupling constant.

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Luckily, this never happens for $h, T \in \mathbb{R}$.

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In these variables, $Z_n(z,t)$ becomes a (Laurent) polynomial in $z$ for each $t$. Easier to handle than exponential functions.

# Thermodynamic quantities in terms of zeros of $Z_n(z, t)$.

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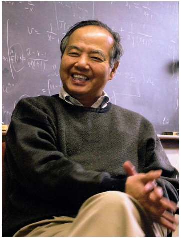

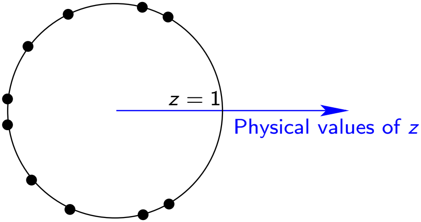

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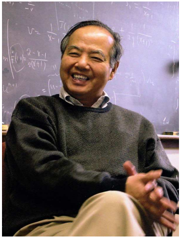

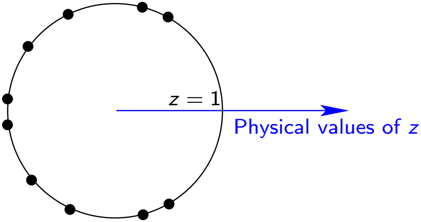


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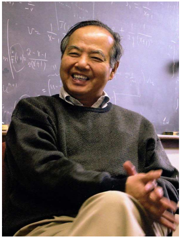

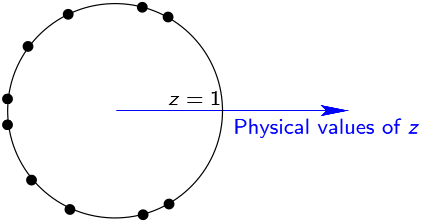


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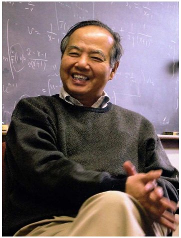
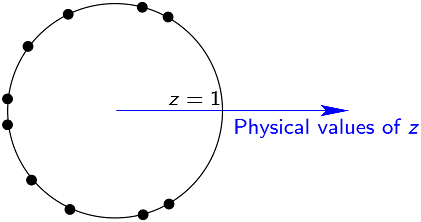




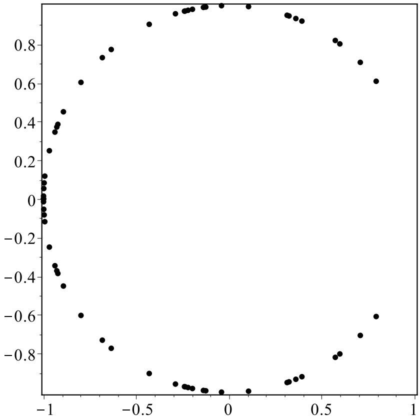

Extensions of this theorem are contemporary mathematics:

D. Ruelle. Characterization of Lee-Yang polynomials. *Ann. of Math.* (2010).

J. Borcea and P. Brändén The Lee-Yang and Pólya-Schur programs. I. Linear operators preserving stability. *Invent. Math.* (2009).

# Numerical Example of Lee-Yang zeros

If Γ is the binary tree 5 generations deep and $t = \frac{1}{2}$, then here are the Lee-Yang zeros:



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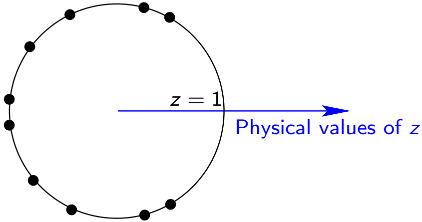

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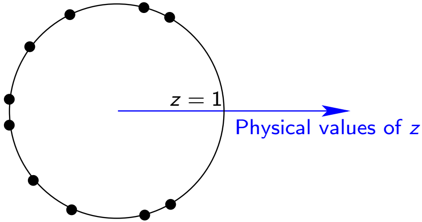


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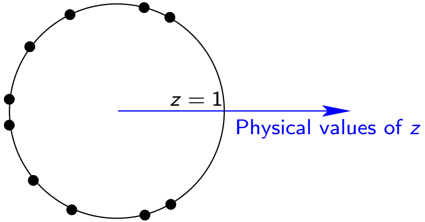


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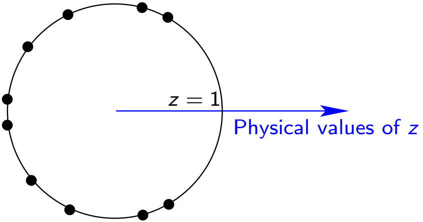


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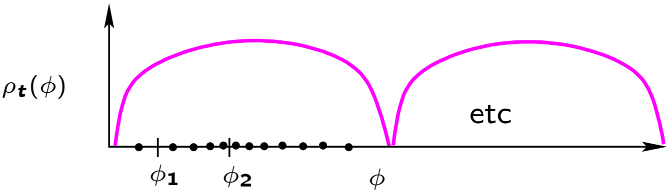

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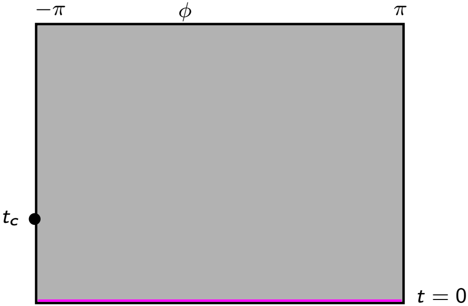
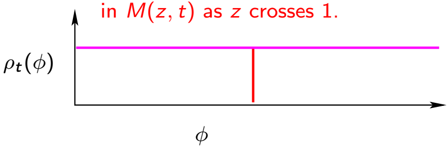
Understanding how the Lee-Yang distributions $\mu_t(\phi)$ vary with $t$ and $\phi$ is essential to understanding phase transitions of the model.

# Expected limiting distributions of Lee-Yang zeros for $\mathbb{Z}^2$




$\rho_t(0) > 0$ gives jump discontinuity in $M(z, t)$ as $z$ crosses 1.



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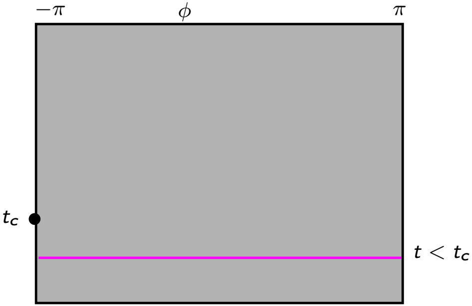


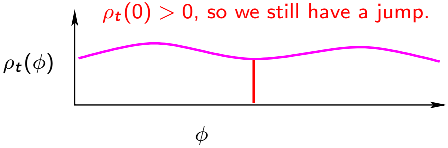

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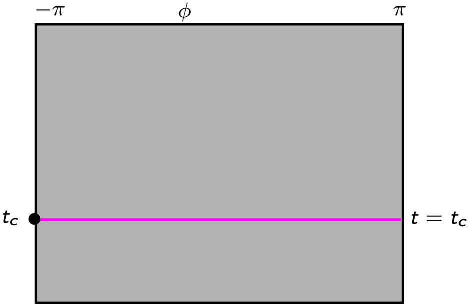
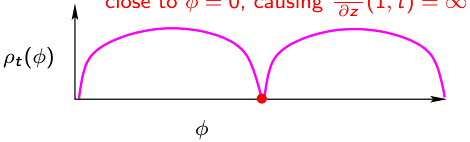



Now, $\rho_t(0) = 0$, so no jump.
However, they accumulate arbitrarily close to $\phi = 0$, causing $\frac{\partial M}{\partial z}(1, t) = \infty$



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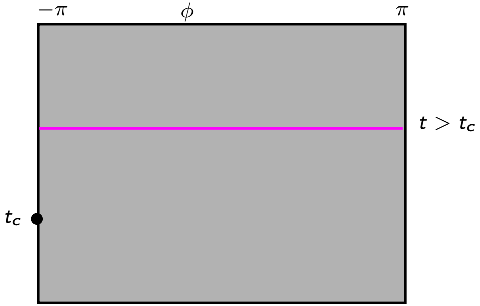
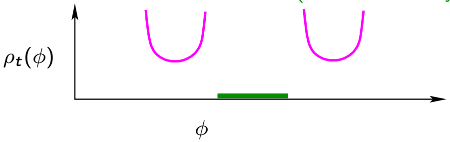



Now, we have a nice interval around $\phi = 0$ with $\rho_t(\phi) \equiv 0$. Causes $M(z,t)$ to be differentiable at $z = 1$ (and hence everywhere).



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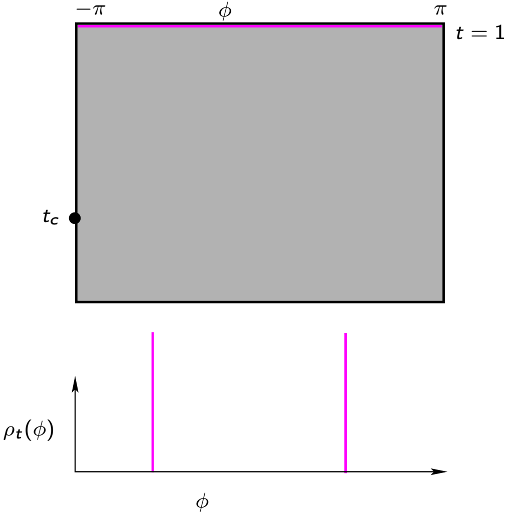

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At the boundary of where $\rho_t(\phi) > 0$, it blows up with exponent $-1/2$.

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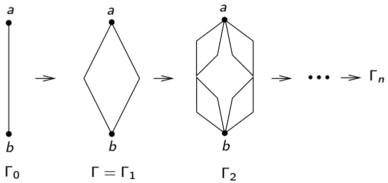

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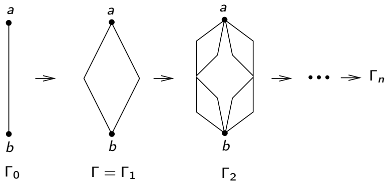


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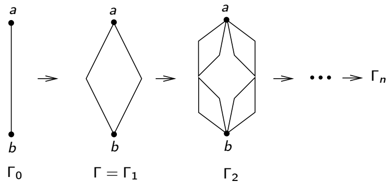


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Pavel Bleher, Mikhail Lyubich, and Roland Roeder. *Lee-Yang Zeros for the DHL and 2D Rational Dynamics, I. Foliation of the Physical Cylinder.* To appear in Journal de Mathématiques Pures et Appliquées (104 pages).

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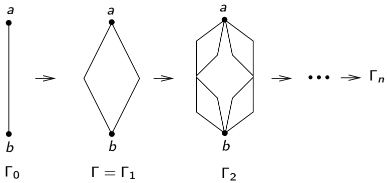


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http://www.math.iupui.edu/~rroeder/Lee_Yang_PartI.pdf.

# What types of mathematics are used when studying the Ising model?

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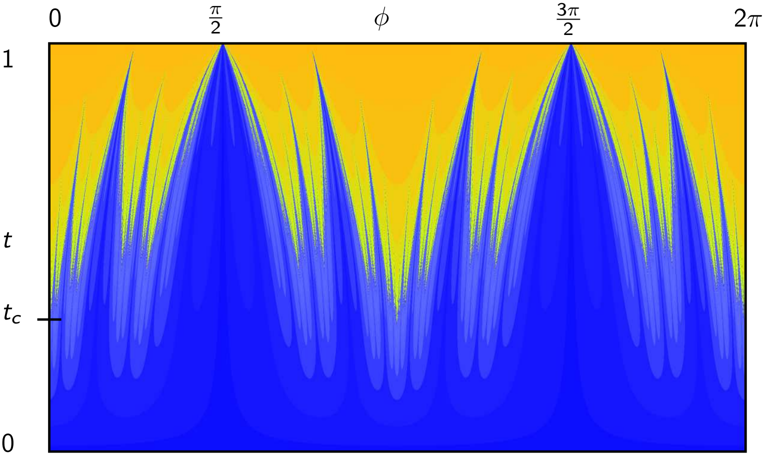
- Complex Analysis
- Real Analysis
- Dynamical Systems
- Algebraic Geometry
- Combinatorics




Thank you for listening!